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On non-archimedean diophantine approximations

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Let $\mathbb{F}_q$ be a finite field of q elements and consider the following :

$\mathbb{F}_q[X]$: the ring of polynomials with $\mathbb{F}_q$-coefficients,

$\mathbb{F}_q(X)$: the fraction field of $\mathbb{F}_q[X]$,

$\mathbb{F}_q((X^{-1}))$: the field of formal Laurent power series with $\mathbb{F}_q$-coefficients.

We denote by $\mathbb{L}$ the set of elements in $\mathbb{F}_q((X^{-1}))$ of negative degree, where we regards

$$0 = 0X^{-1} + 0X^{-2} + 0X^{-3} + \cdots \qquad \in \mathbb{L}, \quad \deg 0 = -\infty.$$

We define

$$|f| = q^{\deg f} = q^n$$

if

$$f = a_n X^n + a_{n-1} + \cdots, \qquad \text{with } a_n \neq 0 \in \mathbb{F}_q$$

Since $\mathbb{L}$ is a compact abelian group with the metric $d(f,g) = |f-g|$ for f, $g \in \mathbb{L}$, there exists the unique normalized Haar measure which we denote by m. In the sequel we consider the following diophantine inequality for $f \in \mathbb{L}$:

$$|f - \frac{P}{Q}| < \frac{\Psi(Q)}{|Q|}, \qquad (P,Q) = 1, \quad P, Q \in \mathbb{F}_q[X] \tag{1}$$

where Ψ is a non-negative function defined on the set of positive integers. Since $|Q| = q^{\deg Q}$ we think Ψ is a $\{q^n : n \in \mathbb{Z}\} \cup \{0\}$-valued function. Our questions are as follows.

(i) (1) has infinitely many solutions $\frac{P}{Q}$ for m-a.e. $f \in \mathbb{L}$ or not.

(ii) If the answer to (i) is yes, then the law of large numbers holds or not.
(iii) If the answer to (ii) is yes, then the central limit theorem holds or not.

In 1970, de Mathan [2] proved that (1) has infinitely many solutions if $q^n\Psi(n)$ is monotone non-increasing and $\sum_{n=1}^{\infty} q^n\Psi(n) = \infty$. It is easy to see that (1) has at most finitely many solutions if $\sum_{n=1}^{\infty} q^n\Psi(n) < \infty$. This means that the 0-1 law holds if $q^n\Psi(n)$ is monotone non-increasing. However, this monotonicity condition is a bit strong. Actually, recently Inoue and Nakada [5] showed the following.

Theorem 1. *(1) has infinitely many solutions P, Q for m-a.e. f if and only if*

$$\sum_{n=1}^{\infty} q^n\Psi(n) = \infty.$$

In the proof of Theorem 1, [5] used the following property. Let $l_1, l_2, \ldots$ be a sequence of non-negative integers. We put

$$F_n = \{f \in \mathbb{L} : \left|f - \frac{P}{Q}\right| < \frac{1}{q^{2n+l_n}},\ (P,Q) = 1,\ \deg Q = n \text{ for some } Q \in \mathbb{F}_q[X]\}.$$

Note that here we think $\Psi(n) = \frac{1}{q^{n+l_n}}$.

Proposition 1. *For $0 < n < m$, we see*

$$m(F_n \cap F_m) = \begin{cases} m(F_n) \cdot m(F_m) & \text{if } m > n + l_n \\ 0 & \text{otherwise.} \end{cases}$$

By this property, we have the strong law of large numbers by using the quantitative Borel-Cantelli lemma (see Philipp [6]). We put

$$W(N) = \sharp\{n : 1 \le n \le N,\ \exists P, Q \in \mathbb{F}_q[X] \text{ s.t. } \left|f - \frac{P}{Q}\right| < \frac{\Psi(\deg Q)}{|Q|},\ (P,Q) = 1\}$$

and

$$Z(N) = \sum_{n=1}^{N} q^n\Psi(n)(1 - \frac{1}{q})$$

Theorem 2.

$$W(N) = Z(N) + O(Z^{1/2}(N)\log^{3/2+\varepsilon} Z(N)) \text{ as } N \to \infty \quad \text{for } m\text{-a.e.} f.$$

The next question is to find a sufficient condition on Ψ for which the central limit theorem holds. About this problem, Fuchs [3] showed that if $\sum_{n=1}^{\infty} q^n\Psi(n) = \infty$, $q^n\Psi(n)$ is monotone non-increasing, and some additional conditions hold, then the central limit theorem holds. Since his proof is based on the stochastic property of continued fraction expansions over $\mathbb{F}_q[X]$, the additional conditions were necessary. However, it is possible to prove the central limit theorem without using continued fractions for this problem. The idea is to generalize Proposition 1 and to construct a non-stationary 1-dependent process from the indicator function of F_n. Actually it is possible to show the following.

Proposition 2. *For* $0 < n_1 < n_2 < \cdots < n_k$, *we have*

$$m(\bigcap_{i=1}^{k} F_{n_i}) = \begin{cases} \prod_{i=1}^{k} m(F_{n_i}) & \text{if } n_i + l_{n_i} < n_{i+1} \text{ for } 1 \le i \le k-1 \\ 0 & \text{otherwise.} \end{cases}$$

We put $n_1 = 1$,

$$G_1 = F_1 \cup F_2 \cup \cdots \cup F_{1+l_1} \quad \text{and} \quad n_2 = 1 + l_1 + 1 = n_1 + l_{n_1} + 1$$

and

$$G_k = F_{n_k} \cup F_{n_k+1} \cup \cdots \cup F_{n_k+l_{n_k}} \quad \text{and} \quad n_{k+1} = n_k + l_{n_k} + 1.$$

Then from Proposition 2, we see that the sequence of the indicator functions 1_{G_k}, $k \ge 1$ is a 1-dependent process. By using this 1-dependent process, we can prove the following theorem.

Theorem 3. *If* $\sum_{n=1}^{\infty} q^n \Psi(n) = \infty$ *and* $q^n \Psi(n)$ *is monotone non-increasing, then the central limit theorem holds, that is,*
(i) if $\lim_{n\to\infty} q^n \Psi(n) = q^{-l}$ *for some positive integer* l, *then*

$$\lim_{N\to\infty} m\{f \in \mathbb{L} : \frac{W(N) - Z(N)}{\sqrt{N}} < \alpha\} = \int_{-\infty}^{\alpha} e^{\frac{-x^2}{2A}} dx$$

where $A = (\frac{1}{q^l}(1 - \frac{1}{q})) - (2l+1)(\frac{1}{q^l}(1-\frac{1}{q}))^2$,
(ii) if $\lim_{n\to\infty} q^n \Psi(n) = 0$, *then*

$$\lim_{N\to\infty} m\{f \in \mathbb{L} : \frac{W(N) - Z(N)}{\sqrt{Z(N)}} < \alpha\} = \int_{-\infty}^{\alpha} e^{\frac{-x^2}{2}} dx.$$

Remark. In the case of (i) in the above, we note that

$$Z(N) \sim N \cdot q^{-l}(1 - \frac{1}{q}).$$

The non-increasingness condition can be weakened. Indeed we can prove the same result for the following function Ψ:

$$\Psi(n) = \begin{cases} \frac{1}{q^{n+l_m}} & \text{if } n = n_m \\ 0 & \text{otherwise,} \end{cases}$$

where $n_1 < n_2 < \cdots < n_m < \cdots$ and $l_1 \leq l_2 \leq \cdots \leq l_m \leq \cdots \nearrow \infty$ are sequences of positive integers and $\sum_{m=1}^{\infty} q^{l_m} = \infty$.

The rest of this paper is for more general case of Ψ's. We consider the inequality

$$\left| f - \frac{P}{Q} \right| < \frac{\Psi(Q)}{|Q|}, \qquad (P,Q) = 1 \tag{2}$$

where Ψ is a non-negative function defined on $\mathbb{F}_q[X]$ and $\Psi(Q) = \Psi(Q')$ whenever $Q' = aQ$ for some non-zero $a \in \mathbb{F}_q$. The main question is to find a necessary and sufficient condition on Ψ having infinitely many solutions $\frac{P}{Q}$ for m-a.e. $f \in \mathbb{L}$. If $\sum_{Q:monic} \Psi(Q) < \infty$, then it is easy to see from the Borel-Cantelli lemma that there exist at most finitely many solutions $\frac{P}{Q}$ for m-a.e. $f \in \mathbb{L}$. Moreover we have the following non-archimedean version of Gallagher theorem.

Theorem 4. (Inoue-Nakada [5]) *For any Ψ, the set of $f \in \mathbb{L}$ having infinitely many solutions $\frac{P}{Q}$ of (2) is either m-measure 0 or 1.*

By using this theorem we can prove a non-archimedean version of Duffin-Schaeffer theorem, which gives a sufficient condition on Ψ for having infinitely many solutions of (2) for m-a.e. $f \in \mathbb{L}$.

Theorem 5. (Inoue-Nakada [5]) *Let Ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function which satisfies*

$$\sum_{n=1}^{\infty} \sum_{\substack{\deg Q = n \\ Q:monic}} \Psi(Q) = \infty.$$

Suppose there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q \leq n \\ Q:monic}} \Psi(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:monic}} \Psi(Q) \frac{\Phi(Q)}{|Q|}$$

holds for a constant C. Then

$$\left| f - \frac{P}{Q} \right| < \frac{\Psi(Q)}{|Q|}, \qquad (P,Q) = 1$$

has infinitely many solutions $\frac{P}{Q}$ for a.e. $f \in \mathbb{L}$.

Remark. (i) It would be natural to ask whether the Duffin-Schaeffer conjecture is true or not in this case, that is, the following condition is a sufficient condition for having infinitely many solutions of (2) for m-a.e. $f \in \mathbb{L}$ or not:

$$\sum_{Q:monic} \Psi(Q) \frac{\Phi(Q)}{|Q|} = \infty$$

where $\Phi(Q)$ denotes the number of $P(\neq 0) \in \mathbb{F}_q[X]$ such that $\deg P < \deg Q$ and $(P, Q) = 1$.
(ii) There is a higher dimension version (the simultaneous approximations result) of this theorem, see [4]. However, in the case of the simultaneous approximations, it seems to be not so easy to find a necessary and sufficient condition for having infinitely many solutions a.e. even in the case that the function "Ψ" depends only on degree of the denominator Q.

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